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Absorbing Markov and branching processes with instantaneous resurrection

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A Markov branching process with instantaneous immigration from the zero state can be constructed so as to be honest and have the non-negative integers as state-space, but the construction requires the branching part to be explosive. We show that a realistic model can be constructed without this restriction if the state-space is restricted to the natural numbers. Moreover this construction is the weak limit, in the sense of finite dimensional laws, of the Yamazato model as the zero state holding-time parameter tends to infinity.

This idea of immediate resurrection from an absorbing subset is extended to any minimal discrete-state Markov process, and even to a larger class. Our emphasis is on existence and uniqueness of the transition functions of the resurrected process, and classification of its states.

branching and Markov processes * transition functions and generators * resurrection * recurrence classification

1. Introduction

Let (Z_t) denote a Markov branching process (MBP) with per capita birth rate $\nu > 0$ and offspring distribution $\{p_j\colon j \geq 0\}$. A good general reference is Athreya and Ney (1972). We assume $0 < p_0 < 1$ and, without loss of generality, $p_1 = 0$; see Pakes (1987, p. 310). Let $f(s) = \sum_{j\geq 0} p_j s^j$, and let q be the least positive solution of $f(s) = s$. A necessary and sufficient condition that (Z_t) be honest is that for each $q < \varepsilon < 1$,

$$\int_{1-\varepsilon}^{1} \mathrm{d}s/(f(s)-s) = -\infty. \tag{1.1}$$

We usually assume this is satisfied, and it is if $m = \sum jp_j < \infty$.

Yamazato (1975) considered a modification (Y_t) of the MBP which allows it to be resurrected whenever it hits the zero state. Specifically, let $\lambda > 0$ and $\{h_j\colon j \geq 1\}$ be a discrete law. If the process hits 0, it sojourns there for a time having an exponential law whose mean is $1/\lambda$ and it then jumps to state j with probability h_j. This resurrection event is independent of the history of the process up to the time it first hits 0, and the sojourn at 0 and the jump size into the positive states $\mathscr{T}$

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are independent. The future evolution of the process is conditionally independent, given the state from which it resurrects. A more careful construction of such a *return process* is given by Pakes and Tavaré (1981), but see below.

Yamazato derived properties of this MBP with immigration from the zero state (MBPI) assuming $m<\infty$, but most of what he does requires (1.1) only. See Pakes (1979) and Pakes and Tavaré (1981) for some other properties.

Recently Chen and Renshaw (1990) addressed the problem of constructing a version of the MBPI which allows instantaneous resurrection from 0. Thus they sought a Markov process satisfying the following conditions:

(a) the state space is $\mathbb{N}_+$;

(b) its generator $[q_{ij}]$ satisfies

$$q_{ij}=u_{ij}\quad (i\geqslant 1,\, j\geqslant 0)$$

where $u_{ij}=\nu i p_{j-i+1}$ $(i\neq j)$, $=-\nu i$ $(i=j)$ defines the generator of the MBP; and

(c) $\quad q_{00}=-\infty \quad$ and $\quad q_{0j}\neq 0 \quad (j\geqslant 1).$

They proved that an honest such process exists only if

$$\sum_{j\geqslant 1} q_{0j}=\infty, \tag{1.2}$$

and when this condition is satisfied exactly one such honest process exists iff, for $\theta>0$,

$$\sum_{i\geqslant 1}\sum_{j\geqslant 0} q_{0i}\hat{r}_{ij}(\theta)<\infty \tag{1.3}$$

where $\hat{r}_{ij}(\theta)=\int_0^\infty r_{ij}(t)\exp(-\theta t)\,\mathrm{d}t$ and $[r_{ij}(t)]$ is the transition matrix of (Z_t). Furthermore, it is clear that (1.2) and (1.3) can hold only if $\sum_{j\geqslant 0} r_{ij}(t)<1$, *i.e.*, (Z_t) is dishonest. They show also that starting with any dishonest MBP it is possible to choose the q_{0j} so that (1.2) and (1.3) hold.

The MBP with instantaneous resurrection is denoted by MBPII (i.e., instantaneous immigration). Chen and Renshaw (1993) have shown that the MBPII is always recurrent, and they give a necessary and sufficient condition for positive recurrence. We stress that the proof of existence of this process is entirely in terms of a construction of the resolvent of a transition semi-group satisfying conditions (a)–(c). There are infinitely many dishonest transition semi-groups satisfying these conditions. Moreover there is no attempt to elucidate the sample path behaviour of the MBPII, which we denote by (C_t).

Intuition suggests that if (C_t) starts from a positive state then there is a positive probability that it will explode through $\mathcal{T}$ before hitting 0. There is no specific mechanism for returning to $\mathcal{T}$, though this must occur since the process is honest, recurrent even. General theory informs us that if $C_t=0$ then as $s\downarrow t$ (or $s\uparrow t$), a.s. C_s has exactly two limits, 0 and ∞. If $S_0=\{t\colon C_t=0\}$ then S_0 contains no open interval but is dense in itself and $P_i(|S_0|>0)>0$. See Chung (1967, Sections II.5, II.6). Moreover $E_i|S_0|=\int_0^\infty p_{i0}(t)\,\mathrm{d}t$. By recurrence this is infinite when $i=0$, and also for $i \geqslant 1$ from Chen and Renshaw (1993, Lemma 2.3) and a calculation in the proof of their Theorem 3.1.

Consequently, although Chen and Renshaws' construction has considerable mathematical interest, we believe that it fails as a realistic model of a branching population which allows instantaneous regeneration from extinction. (These authors do not claim any practical realism.) The problem lies in insisting that the state space includes 0. An MBP about to hit 0 has only a single extant individual, i.e., 0 is accessible from 1 only. At the end of its life this individual either has $j \geqslant 2$ offspring with probability p_j, or it has no offspring with probability p_0. In the latter event heavenly intervention immediately provides $j \geqslant 1$ replacement individuals with probability h_j. Consequently the pragmatic modeller's viewpoint would be that the process never hits 0; its essential state space is $\mathbb{N}$.

Hence we define the (pragmatic) MBPII as any Markov process satisfying:

(A) its state space is $\mathbb{N}$;

(B) its generator is

$$q_{ij} = u_{ij} \quad (i \geqslant 2, j \geqslant 1) \quad \text{and} \quad q_{1j} = u_{1j} + \nu p_0 h_j \quad (j \geqslant 1). \tag{1.4}$$

Chen and Renshaw (1990) assert that it is not possible to allow $\lambda \to \infty$ in Yamazato's model. We show the reverse is true; indeed we will prove weak convergence of the finite dimensional laws to those of a Markov process (X_t) having the generator (1.4). We carry out this programme in the next section.

In Section 3 we pursue the recurrence classification of (X_t), comparing it with those for (Y_t) and (C_t), and we obtain and compare the limiting laws for all three processes. We note that the limit law of (Y_t) was given in unpublished work of Stewart (1976).

We then turn to the question of resurrecting a general Markov process (Z_t) (with a countable state space $\mathscr{S}$) at the instant it hits an absorbing subset H, assumed to be accessible from the transient set $\mathscr{T} = \mathscr{S} \backslash H$. In the existing literature on this topic, reviewed in Section 4, (Z_t) is assumed to be minimal and honest. Our development allows greater generality, the main restriction being that if (Z_t) can escape in finite time through $\mathscr{T}$ to the boundary '∞', then it returns instantaneously to $\mathscr{T}$. Connections are found between the generators of (Z_t) and the resurrected process (X_t) (whose state space is $\mathscr{T}$) and between their transition functions. In particular the transition functions of (X_t) solve a system of Markov renewal equations ((4.8) below), and this system is satisfied by solutions of the backward system of (X_t) iff the transition functions of (Z_t) satisfy their forward system.

Questions of the unique solution of (4.8) are examined in Section 5. The results there are easy consequences of existing Markov renewal theory, and we give some examples showing the limits of our general theory. Finally, in Section 6 we give results on the state classification of (X_t) and identify the limiting law when it exists. Some examples are given, and in particular we show that the general theory is strong enough to give the recurrence classification in Section 3 for the resurrected MBP.

Occasionally some notation is duplicated, but this should not confuse alert readers.

2. The limiting form of Yamazato's model

With the above notation, let $a(s)=\nu(f(s)-s)$, $E_i(\cdot)=E(\cdot|Z_0=i)$ and $F(s,t)=E_1(s^{Z_t})$. Here (Z_t) denotes the minimal process constructed from $[u_{ij}]$ (Harris, 1963). Then

$$E_i(s^z t)=(F(s,t))^i$$

and

$$\partial F/\partial t=a(F), \qquad F(s,0)=s. \tag{2.1}$$

In essence (2.1) is the backward equation satisfied by the $r_{1j}(t)$.

Now let (Y_t) be the Yamazato (1975) process, i.e. the return process constructed from (Z_t) as in Pakes and Tavaré (1981). For our purposes it is best to use the following specific construction; see also Section 4 below. We begin with the construction of the MBP as a randomised left-continuous random walk as described in Athreya and Ney (1972, p. 118). Diagonal allocation of the increment random variables and the elements of a sequence of unit exponential random variables allows us to construct, on the same sample space, an independent sequence of MBP excursions $\{(Z_t^{(n)})\colon n\geqslant 0\}$ such that $Z_0^{(0)}=i$ and $Z_0^{(n)}$ $(n\geqslant 1)$ has the resurrection law $\{h_j\}$. In addition we can construct an independent sequence $\{U_n\colon n\geqslant 1\}$ having the unit exponential law, and so as to be independent of the excursions.

Construct (Y_t) by aligning the excursions in order along the time axis and alternating with the random variables $\{U_n/\lambda\}$. The construction ends with the first excursion not hitting zero, or else it continues forever. Thus U_n/λ separates $(Z_t^{(n-1)})$ and $(Z_t^{(n)})$ if the former hits zero. It is clear that (Y_t) is a Markov process and its transition probabilities are derived by Yamazato (1975). He assumes $m<\infty$ but his results are valid also when $m=\infty$. Note that (Y_t) is honest iff (Z_t) is honest. We are now able to prove the following result.

Theorem 2.1. *Let (Z_t) be the minimal MBP and (Y_t) be the corresponding Yamazato process. Then for each $n\in\mathbb{N}$ and $0\leqslant t(1)<t(2)<\cdots<t(n)$,*

$$(Y_{t(1)},\ldots,Y_{t(n)})\Rightarrow(X_{t(1)},\ldots,X_{t(n)})$$

where (X_t) is a Markov process with generator (1.4) *and whose transition semigroup is determined by* (2.2)–(2.5) *and satisfies the forward and backward Kolmogorov equations. Moreover (X_t) is honest iff (Z_t) is honest.*

Proof. The convergence assertion holds a.s. for our construction of (Y_t) and the limit process (X_t) is just the ordered concatenation of the above MBP excursions, the whole taken to be right-continuous. Hence the zero state is inaccessible, holding times in the positive states have exponential laws, and as $h\to 0+$,

$$P(X_{t+h}=j\,|\,X_t=1)=\nu h(p_0h_j+p_j)+\mathrm{o}(h).$$

It follows that (X_t) is a Markov process with the generator (1.4).

Let $p_{ij}(t)$ $(i, j \in \mathbb{N})$ be the transition probabilities of (X_t) and $\phi_j(t) = \sum_{i \geq 1} h_i p_{ij}(t)$, the probability of reaching j by $t + \tau$ from an extinction-resurrection event at τ. Observing that $r_{i0}(t)$ is the d.f. of the hitting time of zero by the initial MBP excursion, we have

$$p_{ij}(t) = r_{ij}(t) + \int_0^t \phi_j(t-u)\,\mathrm{d}r_{i0}(u)$$

which in p.g.f. form is (with obvious notation)

$$P_i(s,t) = (F(s,t))^i - (F(0,t))^i + \int_0^t \phi(s, t-u)\,\mathrm{d}r_{i0}(u). \tag{2.2}$$

Summing over the return law gives the renewal equation

$$\phi(s,t) = h(F(s,t)) - h(F(0,t)) + \int_0^t \phi(s, t-u)\,\mathrm{d}F(u) \tag{2.3}$$

where $F(t) = h(F(0,t))$ is a (possibly defective) d.f. The corresponding renewal function is

$$V = \sum_{n \geq 0} F^{n*} = 1 + F * V \tag{2.4}$$

and the solution of the renewal equation is

$$\begin{aligned}\phi(s,t) &= \int_0^t [h(F(s,t-u)) - h(F(0,t-u))]\,\mathrm{d}V(u)\\ &= 1 - \int_0^t [1 - h(F(s,t-u))]\,\mathrm{d}V(u).\end{aligned} \tag{2.5}$$

Our construction endows (X_t) with the Feller minimality enjoyed by the MBP excursions, and hence both Kolmogorov systems are satisfied, and (X_t) clearly is honest iff (Z_t) is. All these process-distributional properties are shared by any other (minimal) construction of (X_t). □

This result also gives a non-trivial example in which process convergence is not determined by the limiting q-matrix; cf. Ethier and Kurtz (1986, Problem 8, p. 262). The limiting q-matrix of the Yamazato processes is precisely that considered by Chen and Renshaw (1990) and there are infinitely many processes with this q-matrix.

3. Recurrence classification

Not surprisingly, (X_t) has the same recurrence classification as the Yamazato process though, of course, the limiting stationary laws (LSL's) differ. We collect results in the following:

Theorem 3.1. (a) *The process* (X_t) *is recurrent if* $m \leqslant 1$, *and transient otherwise.*

(b) *When* $m \leqslant 1$, (X_t) *is positive recurrent iff*

$$1/c = \int_0^1 [(1-h(s))/(f(s)-s)]\,\mathrm{d}s < \infty \tag{3.1}$$

and then the LSL $(\pi_j\colon j \in \mathbb{N})$ *has the p.g.f.*

$$\pi(s) = c\int_0^s [(1-h(u))/(f(u)-u)]\,\mathrm{d}u.$$

Proof. Assertion (a) is obvious from our construction—a.s. there is an excursion which drifts (or explodes) to infinity iff $m > 1$.

With $m \leqslant 1$, let

$$\mathcal{I} = \int_0^\infty [h(F(s,t)) - h(F(0,t))]\,\mathrm{d}t = I(1) - I(2)$$

where $I(2) = \int_0^\infty [1-h(F(s,t))]\,\mathrm{d}t$. Using the substitution $u = F(s,t)$, (2.1) yields

$$I(2) = \int_s^1 \frac{1-h(u)}{a(u)}\,\mathrm{d}u,$$

and a similar tratment applied to $I(1)$ yields

$$\mathcal{I} = \int_0^s \frac{1-h(u)}{a(u)}\,\mathrm{d}u.$$

Now let $0 < s < 1$ and choose τ so large that $s < F(0,\tau)$. Then (2.5) and (2.4) yield

$$P(s,t) \leqslant \int_0^t [h(F(0,t+\tau-u)) - h(F(0,t-u))]\,\mathrm{d}V(u)$$

$$= V(t+\tau) - V(t) - \int_t^{t+\tau} F(t+\tau-u)\,\mathrm{d}V(u) \leqslant V(t+\tau) - V(t),$$

where we now denote $h(F(0,t))$ by $F(t)$. But $1/c = \int_0^\infty (1-F(t))\,\mathrm{d}t$, and if this is infinite then Blackwell's theorem yields $\lim_{t\to\infty} P(s,t) = 0$. Conversely, if $1/c < \infty$ then the key renewal theorem and (2.5) yield $\lim_{t\to\infty} P(s,t) = c\mathcal{I}$. It is clear now that $\lim_{t\to\infty} P_i(s,t) = \lim_{t\to\infty} P(s,t)$, and the assertion follows. □

Computing I as above when $m > 1$ yields a generating function for the Green's functions $G_{ij} = \int_0^\infty p_{ij}(t)\,\mathrm{d}t$:

$$\sum_{j\geqslant 1} G_{ij}s^j = \int_0^s [(q^i - u^i)/a(u)]\,\mathrm{d}u$$

$$+ \left\{q^i \int_0^s [(h(q)-h(u))/a(u)]\,\mathrm{d}u\right\} \Big/ \{1-h(q)\}. \tag{3.2}$$

It is interesting to compare $\pi(s)$ above with the corresponding quantities for the Yamazato and the Chen–Renshaw models. First we pause to give an elementary proof of Yamazato's (1975) classification. Let $T(0)$ be the hitting time of 0. If $Y_0=0$ then $T(0)=\varepsilon(0)+\tau(0)$ where $\varepsilon(0)$ is the holding time in 0 and has the $\exp(\lambda)$ law, $\tau(0)$ is the subsequent time to return to 0 and has the d.f. $F(t)$, and the two are independent. Consequently (Y_t) is recurrent iff $\tau(0)<\infty$ a.s. and this holds iff $F(0,1)\to 1$ $(t\to\infty)$, i.e., iff $m\leqslant 1$. Assume this. Then (Y_t) is positive recurrent iff $E\tau(0)=\int_0^\infty (1-F(t))\,\mathrm{d}t<\infty$, i.e., iff (3.1) holds.

Neither Yamazato (1975) nor Chen and Renshaw (1993) determined the LSL's of their models. Stewart (1976) found for (Y_t) the p.g.f.

$$\pi_Y(s)=\pi_{Y0}\left[1+\frac{\lambda}{\nu}\int_0^s \frac{1-h(s)}{f(u)-u}\,\mathrm{d}u\right] \quad \text{where } \pi_{Y0}=1/(1+\lambda/\nu c).$$

Note that

$$\pi_Y(s)=\pi_0+(1-\pi_0)\pi(s),$$

showing that the LSL of (X_t) is stochastically larger than that of (Y_t). In addition $\pi_Y(s)\to\pi(s)$ as $\lambda\to\infty$.

For (C_t) let $A(s)=\sum_{j\geqslant 0} q_{0j}s^j$. Condition (1.3) ensures this converges if $|s|<1$. Chen and Renshaw (1993) show that (C_t) is always recurrent and it is positive-recurrent iff

$$\int_0^1 [(A(q)-A(s))/(f(s)-s)]\,\mathrm{d}s<\infty. \tag{3.3}$$

This formally reduces to (3.1) when $m\leqslant 1$ and $q_{0j}=\lambda h_j$. Let $\gamma_{ij}(t)$ denote the transition probabilities of (C_t).

Theorem 3.2. *Assume* (1.2) *and* (1.3) *hold. When* (3.3) *is satisfied the LSL of* (C_t) *has the p.g.f.*

$$\Gamma(s)=\gamma_0\left[1+\int_0^s [(A(q)-A(s))/a(u)]\,\mathrm{d}u\right]$$

where γ_0 *is chosen so* $\Gamma(1)=1$.

Remark. Since the integral at (1.1) is finite, (3.3) is satisfied iff for some $\varepsilon\in(q,1)$, $\int_{1-\varepsilon}^1 (A(s)/a(s))\,\mathrm{d}s<\infty$.

Proof of Theorem 3.2. By using Abelian theorems the form of $\Gamma(\cdot)$ can be inferred from Laplace transform identities recorded in Chen and Renshaw (1993). They define $\eta_j(\theta)=\sum_{i\geqslant 1} q_{0i}\hat{r}_{ij}(\theta)$ and $\eta(\theta)=\sum_{j\geqslant 1}\eta_j(\theta)$. Recalling that the MBP is dishonest, define $\sigma(t)=\sum_{j\geqslant 0} r_{1j}(t)$ which solves the backward equation $\sigma'(t)=a(\sigma(t))$ and $\sigma(0)=1$. In addition $\sum_{j\geqslant 0} r_{ij}(t)=(\sigma(t))^i$. Clearly

$$\eta(0)=\int_0^\infty [A(\sigma(t))-A(F(0,t))]\,\mathrm{d}t=\int_0^1 [A(q)-A(s))/a(s)]\,\mathrm{d}s$$

where the last equality is established as above by using the backward equation. But $\theta\hat{\gamma}_{00}(\theta)=1/(1+\eta(\theta))$ and since $\gamma_0=\lim_{\theta\to 0}\theta\hat{\gamma}_{00}(\theta)$ we see that $\gamma_0>0$ iff (3.3) holds, and when it does the form of η_0 follows.

Next, $\hat{\gamma}_{0j}(\theta)=\hat{\gamma}_{00}(\theta)\eta_j(\theta)$, whence

$$\gamma_j=\gamma_0\eta_j(0)=\gamma_0\sum_{i\geq 1} q_{0i}\int_0^\infty r_{ij}(t)\,\mathrm{d}t$$

and hence

$$\sum_{j\geq 1}\gamma_j s^j=\gamma_0\int_0^\infty [A(F(s,t))-A(F(0,t))]\,\mathrm{d}t$$

and this can be reduced, as above, using the backward equation for the MBP to obtain the assertion. □

The integrand in (3.3) always $\to\infty$ as $s\to 1$, whence the LSL has an infinite first order moment. Suppose $q_{0j}=L(j)j^{\delta-1}$ and $p_j=M(j)j^{\beta-1}$ where $0<\beta$, $\delta<1$ and L and M are slowly varying (at infinity). Then (C_t) is null recurrent if $\delta>\beta$, positive recurrent when $\delta<\beta$ and then $\sum_{i>j}\gamma_i=\Lambda(j)j^{\delta-\beta}$, where Λ is slowly varying. This shows that the moment $\sum j^\eta\gamma_j$ is finite iff $\eta<\beta-\delta$. Hence the finitude of moments of order <1 depends on the immigration sequence and the offspring law. These assertions follow from Abelian and Tauberian theorems for power series. With fussy attention to details, one can show that both null and positive recurrence can occur when $\beta=\delta$.

Hence the Chen–Renshaw model predicts large equilibrium population sizes, whereas (X_t) has stationary moment behaviour similar to the Yamazato models, and more in accord with 'biological intuition'.

4. Markov processes with instantaneous resurrection

In this section let (Z_t) be a MP on a countable state-space $\mathcal{S}=\mathcal{T}\cup H$ where $\mathcal{T}\cap H=\emptyset$ and H is absorbing, but accessible from any state in $\mathcal{T}$. We suppose that (Z_t) has the conservative generator $\mathcal{U}=[u_{ij}]$ and transition matrix $[r_{ij}(t)]$. Consequently this satisfies the $\mathcal{U}$-backward system (that is, it satisfies the Kolmogorov backward equations for $\mathcal{U}$, and other uses of this sort of terminology should be obvious), but not necessarily the $\mathcal{U}$-forward system. Also

$$r_{ij}(t)\equiv 0\quad\text{if } i\in H,\ j\in\mathcal{T},\quad\text{and}\quad \sum_{j\in H} r_{ij}(t)>0\quad (i\in\mathcal{T}).$$

Clearly $\mathcal{T}$ is $\mathcal{U}$-transient.

In the literature there are various approaches which can be used to construct a process (X_t), with state-space $\mathcal{T}$, as an instantaneously resurrected version of (Z_t). The construction used here is based on that of Pakes and Tavaré (1981). Where convenient we will use Z_t and $Z(t)$ interchangeably, and similarly for other processes.

Suppose $Z_0 \in \mathcal{T}$ and let T_1 ($\leq\infty$) be the hitting time of H. Set $X_t = Z_t$ for $0 \leq t < T_1$. Let $\{p(i, j), j \in \mathcal{T}\}$ $(i \in H)$ be laws such that $\sum_{j\in\mathcal{T}} p(i, j) \equiv 1$. If $T_1 < \infty$ set $X(T_1) = j$ with probability $p(Z(T_1), j)$. Next, let (Z''_t) be an independent copy of (Z_t) with $Z''_0 = X(T_1)$ and let T_2 be the hitting time of H by (Z''_t). Then set $X(T_1 + t) = Z''_t$ $(0 \leq t < T_2)$, and so on, in the obvious manner. This construction can be formalised along lines used by Arjas and Speed (1975, pp. 177, 8), and it is closely related to Kuczura's (1973) notion of a 'piecewise MP'. Here a Markovian excursion is interrupted after a random time which is conditionally independent of the excursion (not so for us), given its initial state. At the instant of interruption, the process is reset to another state according to a transition matrix, and hence conditionally independent of the past, and the excursion laws are allowed to depend on the resetting state.

Pakes and Tavaré assumed $\mathcal{U}$ is regular, but this is not necessary for the above construction. Several questions arise:

(Q1) Is (X_t) a MP with generator Q given by

$$q_{ij} = u_{ij} + \sum_{k\in H} u_{ik}\, p(k, j)? \tag{4.1}$$

(Q2) If (Z_t) is minimal for $\mathcal{U}$, then is (X_t) minimal for Q?

(Q3) Is Q regular when $\mathcal{U}$ is regular; more generally, is there a 1-1 correspondence between a $\mathcal{U}$-process (Z_t) and induced Q-processes?

Resurrection occurs in independent work of Syski (1977) in a slightly different context. He begins with a process (Y_t) for which $\mathcal{S}$ is irreducible and which is the minimal process corresponding to its regular generator, which in partitioned form on $H \times \mathcal{T}$ is

$$\mathcal{G} = \begin{bmatrix} G(1,1) & G(1,2) \\ G(2,1) & G(2,2) \end{bmatrix}. \tag{4.2}$$

Let $\mathcal{U}(i, j)$ be the corresponding partition elements of $\mathcal{U}$, so $\mathcal{U}(1, 2) = 0$. If $\mathcal{U}$ is regular and $[G(2,1) \,|\, G(2,2)] = [\mathcal{U}(2,1) \,|\, \mathcal{U}(2,2)]$ then (Y_t) is a return process induced by (Z_t). Syski (1977) thinks of H as 'taboo' set in the sense that attempts by (Y_t) to enter H via $i \in H$ result in a resetting to $j \in \mathcal{T}$ with probability $p(i, j)$. Denote the reset, or *modified*, process by (M_t). Its state-space is $\mathcal{S}$ and its generator is

$$\mathcal{M} = \begin{bmatrix} G(1,1) & G(1,2) \\ 0 & Q \end{bmatrix} \tag{4.3}$$

where $Q = G(2,2) + G(2,1)\Pi(1,2)$ and $\Pi(1,2) = [p(i, j)]$. To retain the above-mentioned connection between (Z_t) and (Y_t) we replace G by $\mathcal{U}$ in this expression for Q, that is, Q is defined by (4.1). Thus $\mathcal{T}$ is a closed set for (M_t) and it is accessible from H. The restriction of (M_t) to $\mathcal{T}$ gives the resurrected process (X_t).

Let $\mathcal{M}(t)$ be the transition matrix of (M_t). Syski (1977) defines a compensation kernel $\mathcal{C}(t) = \mathcal{M}(t)(\mathcal{M} - \mathcal{G})$. In some significant cases $\mathcal{C}(t)$ has a simple form, allowing quick proofs of known identities. This is quite well illustrated by Keilson

(1979, p. 52), in the case of a discrete time random walk on $\mathbb{Z}$. A reflecting barrier at the origin is equivalent to resurrection from $H = -\mathbb{N}$ and the compensation kernel has a two-point support. Amongst other things, Syski (1977) shows there is no particular relation between the recurrence classification of (Y_t) and (X_t). His discussion focusses on properties of (M_t), whereas here we are interested in relationships between (Z_t) and (X_t).

Feigen and Rubinstein (1979) give a 'sample path' construction of (X_t) as follows. Assume $\mathcal{U}$ is regular and (Z_t) is the minimal $\mathcal{U}$-process. Let (Y_t) be the particular return process for which $G(1,1)$ is the identity matrix $\mathcal{I}$ and $G(1,2) = \Pi(1,2)$. Alternatively, this can be regarded as a modification of the general irreducible return process in which an excursion in H starting from $i \in H$ and ending with a jump back into $\mathcal{T}$ is replaced by an exp(1) sojourn in i and a jump to $j \in \mathcal{T}$ with probability $p(i, j)$. Define $\hat{\phi}(s) = \int_0^s I_{\mathcal{T}}(Y_u)\,du$ and $\hat{T}(t) = \sup\{s: \hat{\phi}(s) \leq t\}$. Then $X_t = Y_{\hat{T}(t)}$ is a MP on $\mathcal{T}$ with generator Q. The random clock defined by $\hat{T}$ runs at unit rate while $Y_t \in \mathcal{T}$ and stops otherwise, thus concatenating successive excursions in $\mathcal{T}$ in a cádlág manner. This construction and the assertion of Proposition 1 in Feigen and Rubinstein (1979) remain valid under the lesser assumption that (Y_t) is minimal, honest or not.

Next, Feigen and Rubinstein (1979) develop much more clearly a theme of Syski (1977), namely, to find conditions under which there is a function $\gamma: \mathcal{S} \to \mathcal{T}$ such that $\gamma(Y_t)$ is a Q-process. A sufficient condition that this gives an MP is Dynkin's (1965, p. 325) D-condition:

For $i, k \in \mathcal{S}$ such that $\gamma i = \gamma k$ then

(D) $\quad P_i(Y_t \in \gamma^{-1} j) = P_k(Y_t \in \gamma^{-1} j) \quad$ for all $j \in \mathcal{T}$.

They restrict attention to γ satisfying

$$\gamma i = i \quad \text{if } i \in \mathcal{T} \quad \text{and} \quad \gamma i \in \mathcal{T} \quad \text{if } i \in H,$$

and they set $p(i, j) = \delta_{j,\gamma i}$. If this γ satisfies (D) then there is a very simple connection between the $p_{ij}(t)$ and the transition probabilities $\psi_{ij}(t)$ of (Y_t),

$$p_{ij}(t) = \psi_{ij}(t) + \sum_{k \in H} \psi_{ik}(t) p(k, j). \tag{4.4}$$

Feigen and Rubinstein (1979) show that (D) holds iff

(DQ) $\quad \mathcal{V} = \Pi\mathcal{G}\Pi - \mathcal{G}\Pi = 0, \quad$ where $\Pi = \begin{bmatrix} 0 & \Pi(1,2) \\ 0 & \mathcal{I} \end{bmatrix}$,

and that (4.4) holds only if

$$G(2,1)G^n(1,1)\mathcal{V}(1,2) = 0 \quad \text{for all } n \in \mathbb{N}_+. \tag{4.5}$$

Conversely, this implies (4.4) if $\mathcal{G}$ is bounded. Obviously (4.5) holds if (DQ) does.

When (Z_t) is a MBP then $G(1,1) = -\lambda$ and $G(2,1)$ is the column $(\nu p_0, 0, 0, \ldots)'$. It follows that (4.5) holds iff $h_1 = 1$ and $p_0 = 1$. Hence (Z_t) is the pure death process and (X_t) is the modification of this which makes the state 1 absorbing. Thus the resurrected process descends one state at a time, in the fashion of the pure death process, until it reaches 1. Then each time it tries to enter state 0 it is returned to 1, i.e., this state becomes absorbing.

We now provide answers to (Q1) and (Q2), leaving (Q3) to the next section. Write $T_H = T_1$ to emphasise the hitting of H, and for $i, j \in \mathcal{T}$ define

$$H_{ij}(t) = P_i(T_H \leqslant t;\ X(T_H) = j). \tag{4.6}$$

At this point we make the following assumption which holds in the sequel:

Assumption I. The boundary state '∞' is either absorbing or instantaneous.

Since H is absorbing, (Z_t) can reach the boundary only through $\mathcal{T}$ or through H. The argument we are about to give requires that (Z_t) either is $\mathcal{U}$-minimal or that it can reach H after explosions to the boundary only by jumping from states in $\mathcal{T}$. Assumption I ensures this. Clearly $t < T_H < t + \mathrm{d}t$ and $X(T_H) = j$ iff for some $k \in \mathcal{T}$ and $l \in H$ we have $Z_t = k$, a single jump to l in $(t, t + \mathrm{d}t)$ (with probability $u_{kl}\,\mathrm{d}t + o(\mathrm{d}t)$), and a replacement to j with probability $p(l, j)$. Hence

$$H_{ij}(t) = \int_0^t \sum_{k \in \mathcal{T},\, l \in H} r_{ik}(v) u_{kl} p(l, j)\,\mathrm{d}v. \tag{4.7}$$

From the definition of the resurrected process, $X_t = j$ iff $Z_t = j$ or if $T_H = v$, $X(T_H) = k \in \mathcal{T}$ and there is a conditionally independent excursion from k to j during (v, t). Consequently we have the following basic relation, defined for $i, j \in \mathcal{T}$,

$$p_{ij}(t) = r_{ij}(t) + \int_0^t \sum_{k \in \mathcal{T}} p_{kj}(t - v)\,\mathrm{d}H_{ik}(v). \tag{4.8}$$

The following results answer (Q1) and (Q2).

Theorem 4.1. (i) *If $[r_{ij}(t)]$ is a $\mathcal{U}$-transition matrix then* (4.8) *has a minimal transition matrix solution $[p_{ij}(t)]$.*

(ii) *Any transition-matrix solution of* (4.8) *has the generator Q (see* (4.1)*), and hence satisfies the Q-backward system.*

(iii) *A solution of the Q-backward system satisfies* (4.8) *iff $[r_{ij}(t)]$ satisfies the $\mathcal{U}$-forward system restricted to $\mathcal{T}$. In particular, if $[r_{ij}(t)]$ is $\mathcal{U}$-minimal then the minimal solution of* (4.8) *is Q-minimal.*

Proof. (i) Equation (4.8) has the same form as (2.1) on p. 67 of Anderson (1991). It is obvious that $H_{ij}(\cdot)$ is non-decreasing and continuously differentiable. Also, Fubini's theorem yields

$$\sum_{k \in \mathcal{T}} r_{ik}(t) H'_{kj}(v) = \sum_{\iota \in \mathcal{T},\, l \in H} r_{i\iota}(t + v) u_{\iota l} p(l, j) = H'_{ij}(t + v).$$

Next,

$$\sum_{j\in\mathcal{T}}(r_{ij}(t)+H_{ij}(t))\leq P_i(T_H>t)+\int_0^t\sum_{k\in\mathcal{T},l\in H}r_{ik}(v)u_{kl}\,\mathrm{d}v$$
$$=P_i(T_H>t)+P_i(T_H\leq t)=1.$$

The assertion now follows directly from Lemma 2.1 of Anderson (1991, p. 67).

(ii) Applying Fubini's theorem and the integral mean value theorem to (4.8) yields

$$p_{ij}(t)=r_{ij}(t)+\sum_{k\in\mathcal{T}}H_{ik}(t)r_{kj}(\zeta_{kj})$$

where $0<\zeta_{kj}<t$. In addition, as $t\to 0$,

$$H_{ik}(t)/t\to\sum_{\iota\in\mathcal{T},l\in H}\delta_{i\iota}u_{\iota l}p(l,k)=\sum_{l\in H}u_{il}p(l,k).$$

Hence Fatou's lemma shows that

$$\Omega_{ij}=p'_{ij}(0)\geq q_{ij}. \tag{4.9}$$

But Q is conservative so $\sum_{j\in\mathcal{T}}\Omega_{ij}\geq 0$, and as this sum cannot be positive (Anderson (1991), p. 12) we must have equality at (4.9). The assertion follows.

(iii) Let $\mathcal{P}$ and $\mathcal{R}$ denote the resolvent matrices of the $p_{ij}(t)$ and $r_{ij}(t)$, respectively. The $\mathcal{U}$-forward inequalities for $r_{ij}(t)$ can be written as the resolvent equality

$$\theta\mathcal{R}(2,2)=\mathcal{I}(2,2)+\mathcal{R}(2,2)\mathcal{U}(2,2)+\mathcal{A} \tag{4.10}$$

where $\mathcal{A}_{ij}$ is the Laplace transform of $r'_{ij}(t)-\sum_{k\in\mathcal{T}}r_{ik}(t)u_{kj}$ $(i,j\in\mathcal{T})$, which is non-negative, and zero iff the $\mathcal{U}$-forward system (restricted to $\mathcal{T}$) is satisfied. Now, left multiplication by $\mathcal{R}(2,2)$ of the backward system for $\mathcal{P}$, and using (4.10), gives

$$\theta\mathcal{R}(2,2)\mathcal{P}=\mathcal{P}+\mathcal{R}(2,2)\mathcal{U}(2,2)\mathcal{P}+\mathcal{A}\mathcal{P}$$
$$=\mathcal{R}(2,2)+\mathcal{R}(2,2)\mathcal{U}(2,2)\mathcal{P}+\mathcal{R}(2,2)\mathcal{U}(2,1)\Pi(1,2)\mathcal{P}$$

or

$$\mathcal{P}+\mathcal{A}\mathcal{P}=\mathcal{R}(2,2)+\mathcal{R}(2,2)\mathcal{U}(2,1)\Pi(1,2)\mathcal{P}.$$

But this is the resolvent version of (4.8) iff $\mathcal{A}=0$, proving the first assertion. □

If $[r_{ij}(t)]$ is $\mathcal{U}$-minimal then the Q-minimal transition functions satisfy (4.8), and hence must be the minimal solution of this system.

Remark. Part (iii) shows that the Q-backward system may have solutions which do not satisfy (4.8), that is, there may be Q-processes which cannot be interpreted as a resurrected $\mathcal{U}$-process. This occurs, for example, if $[r_{ij}(t)]$ is an honest non-minimal solution of the backward system of an explosive MBP, since the forward system is solved only by the minimal transition matrix (Harris, 1963, p. 99).

5. Uniqueness

Suppose $r_{ij}(t)$ $(i, j \in \mathscr{T})$ is given and satisfies the forward equation condition of Theorem 4.1(iii). We might expect, or hope, that (4.8) has a unique solution, or only one that is a transition function. Such uniqueness would show that explosiveness of resurrected processes corresponding to a given Q derives from that of the driving $\mathscr{U}$-process (Z_t), that is, resurrection does not introduce a new source of explosiveness. The general theory and examples below show this is not true in general, though we expect it will be true in most cases of interest where H is a small set; in particular, when $\# H = 1$.

The following result follows from Anderson's lemma, used above.

Lemma 5.1. *If the minimal solution of* (4.8) *is honest, then it is unique.* □

This raises the question of whether honesty of a solution to (4.8) implies that of $[r_{ij}(t)]$. The following result addresses this. Let J_∞ be the time of first infinity of the minimal $\mathscr{U}$-process, whence $\sum_{j\in\mathscr{T}} r_{ij}(t) = P_i(J_\infty > t,\ T_H > t)$, where here the $r_{ij}(t)$ are $\mathscr{U}$-minimal.

Theorem 5.1. *Suppose $r_{ij}(t)$ is the $\mathscr{U}$-minimal function and* (4.8) *has an honest solution.*

(i) *If (Z_t) can escape to '∞' only through $\mathscr{T}$ (i.e., $T_H < J_\infty$ implies $J_\infty = \infty$) then $[r_{ij}(t)]$ is honest.*

(ii) *If (Z_t) can escape only through H then a.s. $J_\infty > T_H$.*

(iii) *If escape through both H and $\mathscr{T}$ are possible then*

$$P_i(J_\infty > T_H) + P_i(J_\infty = T_H = \infty) = 1.$$

Proof. (i) Referring to the above, summing (4.8) over $j \in \mathscr{T}$ yields

$$\begin{aligned} 1 &= P_i(J_\infty,\ T_H > t) + P_i(T_H \leqslant t,\ J_\infty = \infty) \\ &\leqslant P_i(J_\infty,\ T_H > t) + P_i(T_H \leqslant t,\ J_\infty > t) = P_i(J_\infty > t). \end{aligned}$$

Hence a.s. $J_\infty = \infty$.

(ii) This is a tautology since explosion cannot precede entry to H.

(iii) In general we have

$$\begin{aligned} 1 &= P_i(J_\infty,\ T_H > t) + P_i(T_H \leqslant t,\ J_\infty > T_H) \\ &= P_i(t < T_H < J_\infty) + P_i(t < J_\infty < T_H) + P_i(T_H \leqslant t,\ J_\infty > T_H) \\ &= P_i(J_\infty > T_H) + P_i(t < J_\infty < T_H). \qquad \square \end{aligned}$$

Now let $t \to \infty$.

Remarks. 1. Case (i) is of most interest in applications.

2. An example for (ii) showing that $[r_{ij}(t)]$ need not be honest follows by modifying the minimal divergent birth process (Z_t) on $\mathbb{N}$ as follows. Choose a

positive integer M and set $\mathscr{T}=\{1,2,\ldots,M\}$ and $H=\mathbb{N}\backslash\mathscr{T}$. Define (X_t) by choosing $p(i,j)=\delta_{i,M+1}\delta_{1j}$, that is, the resurrected process ascends through $\mathbb{N}$ until it hits $M+1$ when it is reset to state 1. Clearly (X_t) is honest, but not (Z_t) since it explodes through H.

3. Result (iii) says that (X_t) honest implies that a.s. either (Z_t) hits and, possibly, explodes through H, or it stays within $\mathscr{T}$ without exploding.

General questions of uniqueness are best tackled by recognising that (4.8) is a system of Markov renewal equations corresponding to the the semi-Markov kernel $H_{ij}(t)$. Indeed, this belongs to the Markov chain (J_n, T_n, L) (Çinlar, 1969, 1975) where T_n is the epoch of the nth resurrection (so $T_0=0$ and $T_1=T_H$) and $J_n=X(T_n)$, the return state at T_n, and L is the total number of resurrections ever made.

Let $\Phi=\mathscr{R}(2,2)\mathscr{U}(2,1)\Pi(1,2)$ and let $M(t)=[M_{ij}(t)\colon i,j\in\mathscr{T}]$ be the Markov renewal function induced by $H_{ij}(t)$; hence $\int_0^\infty \mathrm{e}^{-\theta t}\,\mathrm{d}M(t)=\sum_{n\geqslant 0}\Phi^n$. Then the most general solution of (4.8) (Çinlar, 1969, p. 137) is

$$p_{ij}(t)=g_{ij}(t)+\int_0^t \sum_{k\in\mathscr{T}} r_{kj}(t-v)\,\mathrm{d}M_{ik}(v)\quad (i,j\in\mathscr{T}) \tag{5.1}$$

where $g_{ij}(t)$ is non-negative, bounded, and satisfies

$$g_{ij}(t)=\int_0^t \sum_{k\in\mathscr{T}} g_{kj}(t-v)\,\mathrm{d}M_{ik}(v). \tag{5.2}$$

If γ is the matrix of Laplace transforms of the g_{ij} then (5.2) is equivalent to $\gamma=\Phi\gamma$. The integral term in (5.1) is the minimal solution of (4.8), and hence is a Q-transition function.

Let N_t be the number of transitions made in $[0,t]$ by the above-mentioned semi-Markov process. Then the n-fold iteration of (5.2) can be written as

$$g_{ij}(t)=\int_0^t \sum_{k\in\mathscr{T}} g_{kj}(t-v)\,\mathrm{d}_v E_i(N_v\geqslant n;\ J_n=k)\leqslant P_i(N_t\geqslant n).$$

If N_t is a.s. finite for each t, then allowing $n\to\infty$ shows that $g_{ij}(t)\equiv 0$; the minimal solution of (4.8) is unique. The following is an obvious corollary of this remark.

Lemma 5.2. *If $P_i(T_H<\infty)<1$ $(i\in\mathscr{T})$ then* (4.8) *has exactly one solution.* $\square$

The one-step transition matrix of the Markov chain (J_n) has elements

$$\begin{aligned} h_{ij}&=P_i(T_H<\infty;\ X(T_H)=j)\\ &=\sum_{l\in H}\int_0^\infty\Big(\sum_{k\in\mathscr{T}} r_{ik}(v)u_{kl}\Big)\,\mathrm{d}v\,p(l,j). \end{aligned} \tag{5.3}$$

The matrix $\mathscr{H}=[h_{ij}]$ is sub-stochastic iff the hypothesis of Lemma 5.2 is satisfied, that is, a.s. $L<\infty$. Clearly, $\mathscr{H}$ is stochastic iff T_H is a.s. finite, and then $[r_{ij}(t)]$ is honest because

$$1=\lim_{t\to\infty}\sum_{j\in H} r_{ij}(t)\le \lim_{t\to\infty}\sum_{j\in\mathscr{S}} r_{ij}(t)\le \sum_{j\in\mathscr{S}} r_{ij}(t);$$

see Anderson (1991, p. 7).

There are many situations in which, with respect to $\mathscr{H}$, $\mathscr{T}$ can be partitioned into a collection of ephemeral states and a closed class, $\mathscr{J}$ say. In the sequel we will assume that for (Z_t) each state in H is accessible from each state in $\mathscr{T}$, and then $\mathscr{J}=\{j\colon p(l,j)>0 \text{ for some } l\in H\}$, which by assumption is non-empty. Again by assumption, resurrection from any state in H is possible. Hence if $\mathscr{H}$ is stochastic, then its restriction to $\mathscr{J}$ is stochastic and $H_{ij}(t)>0$ iff $j\in\mathscr{J}$. For example, $\mathscr{J}=\{a\}$ if $p(l,j)=\delta_{qj}$. A situation of this sort occurs in Pakes (1979).

The following is a well-known criterion for uniqueness; see Çinlar (1969, p. 143).

Lemma 5.3. *If $\mathscr{J}$ is $\mathscr{H}$-irreducible and $\mathscr{H}$-recurrent, then* (4.8) *has exactly one solution.* □

The following result is important in applications. It will be generalised in the next section.

Corollary 5.1. *If $\#H=1$ then* (4.8) *has exactly one solution.*

Proof. Obviously H consists of a single absorbing state, a say, and then from (5.3),

$$h_{ij}=r_{ia}(\infty)p(a,j)$$

and this is independent of i. Consequently (J_n) is an i.i.d. sequence, and hence $\mathscr{J}$ is recurrent. □

In many applications all states in H are absorbing, or $\mathscr{U}$ can be modified to make this so. Then we have

$$H_{ij}(t)=\sum_{l\in H} r_{il}(t)p(l,j),$$

(which also follows formally from (4.6) and the $\mathscr{U}$-forward system). The following criterion then comes from Çinlar (1969, p. 135):

Lemma 5.4. *If* $\sup_{i\in\mathscr{T}}\sum_{j\in H} r_{ij}(t)<1$ *for some $t>0$ then* (4.8) *has exactly one solution.* □

Example 5.1. We now show that resurrection can induce explosions. For each $i\in\mathbb{N}$ specify numbers q_i such that $\sum_{i>0}1/q_i<\infty$, $0<\alpha_i=1-\beta_i<1$, and define $A_i=\prod_{1\le k\le i}\alpha_k$. Let $\mathscr{T}=\mathbb{N}$, $H=-\mathbb{N}$ and define $\mathscr{U}$ by: For $i\in\mathbb{N}$,

$$u_{i,i+1}=q_i\alpha_1,\qquad u_{ii}=-q_i,\qquad u_{i,-i}=q_i\beta_i$$

and $u_{ij}=0$ for all other $i, j \in \mathcal{S}$. For the minimal construction this corresponds to a truncated pure birth process; it climbs from $i \in \mathbb{N}$ to $i+1$ with probability α_i and it jumps to the absorbing state $-i$ with probability β_i. The forward equations

$$r'_{ij}(t) = -r_{ij}(t)q_j + r_{i,j-1}(t)q_{j-1}\alpha_{j-1} \quad (1 \leqslant i \leqslant j)$$

can be solved in the same step-wise fashion as is used for the pure birth process. Consequently the $\mathcal{U}$-forward system is *uniquely* solved by the $\mathcal{U}$-minimal transition functions.

Consider the resurrection scheme $p(-i, j) = \delta_{i+1,j}$ whose effect on the minimal (Z_t) is simply to restore the minimal pure birth process, which here is explosive. Two cases arise. Let $A = \lim_{i\to\infty} A_i$ and observe that if $A > 0$, that is, $\sum \beta_j < \infty$, then $P_i(T_H = \infty) > 0$ and a.s. on $\{T_H = \infty\}$, (Z_t) explodes to infinity. Hence the $\mathcal{U}$-minimal transition function is dishonest. By Lemma 5.2 the corresponding Q-minimal transition function, which is just that of the pure birth process, is the unique solution of (4.8). In particular the non-minimal solutions of the Q-backward system do not satisfy (4.8), the corresponding Q-processes cannot arise from this particular resurrection mechanism.

When $A = 0$, $P_i(T_H < \infty) = 1$ and now the $\mathcal{U}$-minimal transition function is honest. But by Theorem 4.1(ii) the Q-minimal transition function solves (4.8), showing that resurrection can induce explosions in an honest absorbing process. Moreover, by Theorem 4.3(iii) *all* solutions of the Q-backward system, which are legion, solve (4.8).

Example 5.2. Modify the above example as follows. For $i \in \mathbb{N}$ let $u_{i,i+1}$ and u_{ii} be as above, but now let $u_{i,-1} = q_i\beta_i$. Suppose that $q_i = -u_{ii} = u_{i,i-1}$ is defined and positive for all $i \in H$; all other u_{ij} equal zero. Assume $\sum_{i\in\mathbb{N}} 1/q_i = \infty$ and $\sum_{i\in H} 1/q_i < \infty$. Hence if (Z_t) hits H then it enters via -1 and then descends explosively through H as a pure death process. Clearly (Z_t) is dishonest, but (X_t) is honest.

In general, if H consists entirely of absorbing states we let $a(i, j) = P_i(Z(T_H) = j)$ $(i \in \mathcal{T}, j \in H)$. The following example shows that $\mathcal{J}$ can be reducible if the above accessibility conditions are relaxed.

Example 5.3. Referring to Example 5.1, write $A_i^j = A_j/A_{i-1}$ if $j \geqslant i > 0$ and $A_i^{i-1} = 1$. Assume $\sum_{i\geqslant 1} 1/q_i = \infty$. We have $a(i, -l) = 0$ if $l < i$, $= A_i^{l-1} - A_i^l$ if $l \geqslant i$. Then

$$h_{ij} = \sum_{l\geqslant i} a(i, -l)\delta_{l+1,j} = a(i, 1-j),$$

and this is zero if $j < i+1$. Hence $\mathcal{J}$ consists of a nested sequence of closed sets and is reducible.

6. State classification

In this section we consider the state classification of (X_t) when $[p_{ij}(t)]$ is honest. There are three possible approaches to this. First, we can assume that Q is regular and classify the states using only the structure of Q; see Anderson (1991, Section 5.3), and the references he gives, and also Wu (1965). Secondly, in principle the states can be classified relative to any honest Q-transition function obtained from the Q-minimal function via Doob's construction or from an entrance law. See Pollett (1990) for this. Since we are interested in (X_t) as a resurrected process we follow the third option, applying the general limit theory of Markov renewal equations.

For $i, j \in \mathscr{T}$ let $\rho_{ij} = \int_0^\infty r_{ij}(t)\,dt$, which always is finite (because H is accessible and absorbing), $G_{ij} = \int_0^\infty p_{ij}(t)\,dt$, and $\eta_{ij} = \sum_{n\geq 0} h_{ij}^{(n)}$. The interpretation of these quantities as mean occupation times is well known, but we remark that η_{ij} is the mean number of resurrections into j from i.

Consider first the case where $\mathscr{J}$ is $\mathscr{H}$-transient, a case embracing the possibility that $\mathscr{H}$ is strictly substochastic. When $\mathscr{H}$ is strictly substochastic there is a positive probability of only finitely many resurrections and then there is a last excursion of (X_t) through $\mathscr{T}$ which behaves like an H-avoiding excursion of (Z_t); in other words $\mathscr{T}$ is Q-transient.

When $\mathscr{H}$ is stochastic and $\mathscr{T}$ is $\mathscr{H}$-transient it is not in general clear what will be its Q-classification. Taking $A = 0$ in Example 5.3 shows that this situation can occur, albeit without our irreducibility assumptions, and for this example $\mathscr{T}$ is Q-transient since (X_t) has non-decreasing paths. We will restrict attention to the 'typical situation' where H is a finite set of absorbing states, and then (even if $\# H = \infty$)

$$h_{ij} = \sum_{l\in H} a(i, l)p(l, j).$$

Under our standing assumptions we have $h_{ij} > 0$ iff $j \in \mathscr{J}$. In the next result we show for the typical situation that if $\mathscr{H}$ is stochastic then $\mathscr{T}$ is $\mathscr{H}$-recurrent. To state it, fix $a \in \mathscr{J}$ and for $i \in \mathscr{J}$ let $\alpha(i) = P_i(J_n = a \text{ for some } n \geq 1)$, $\alpha = \inf_{i\in\mathscr{J}} \alpha(i)$ and $\mathscr{A} = \{i \in \mathscr{J}: \alpha(i) = \alpha\}$.

Theorem 6.1. *Fix $a \in \mathscr{J}$, suppose $h_{ia} > 0$ for some $i \neq a$, $i \in \mathscr{J}$ and for each $l \in H$ suppose*

$$\sum_{j\in\mathscr{J}} p(l, j)\alpha(j) > 0. \tag{6.1}$$

If $P_i(T_H < \infty) = 1$ $(i \in \mathscr{T})$ then a is recurrent.

Proof. Assume a is transient. For $i \in \mathscr{J}$,

$$\alpha(i) = h_{ia} + \sum_{j\neq a} h_{ij}\alpha(j) \geq \sum_{j\in\mathscr{J}} h_{ij}\alpha(j) \tag{6.2}$$

and since $\alpha(a) < 1$ we see, by choosing i so $h_{ia} > 0$, that $\{\alpha(i): i \in \mathscr{J}\}$ cannot be a constant sequence.

Let $l' \in H$ be the state at which $\sum_{j\in\mathscr{J}} p(l, j)\alpha(j)$ is least; by hypothesis it is positive. Then

$$\alpha(i) \geq \sum_{j\in\mathscr{J}} p(l', j)\alpha(j) \sum_{l\in H} a(i, l),$$

and by hypothesis the inner sum is unity. Consequently $\alpha \geq \sum_{j\in\mathcal{J}} p(l', j)\alpha(j)$, and this certainly is impossible if $\mathcal{A} = \emptyset$.

If $\mathcal{A} \neq \emptyset$, then it is a proper subset of $\mathcal{J}$; otherwise $\{\alpha(i): i \in \mathcal{J}\}$ is a constant sequence. By our assumptions $\alpha(i) > 0$ for $i \in \mathcal{J}$ and hence $\alpha > 0$. Consequently $\alpha(j) > \alpha$ if $j \in \mathcal{J}\backslash\mathcal{A}$, but all this is incompatible with (6.2) when $i \in \mathcal{A}$. Consequently a is recurrent. □

Under light communication conditions the last result entails the Q-recurrence of $\mathcal{T}$ as a consequence of the following generally valid result.

Theorem 6.2. *For $i, j \in \mathcal{T}$,*

$$G_{ij} = \sum_{k\in\mathcal{J}} \eta_{ik}\rho_{kj}. \tag{6.3}$$

Proof. Let $O_j^{(n)} = |\{t: X_t = j, t \leq T_n\}|$ be the occupation time of (X_t) in j up to the nth resurrection, if this occurs, and let $G_{ij}^{(n)} = E_i(O_j^{(n)})$. Then $G_{ij}^{(0)} = 0$ and for $n \geq 1$ the regenerative nature of resurrection yields

$$G_{ij}^{(n)} = \rho_{ij} + \sum_{k\in\mathcal{J}} h_{ik} G_{kj}^{(n-1)}. \tag{6.4}$$

A little more explicitly, since $T_1 = T_H$, $O_j^{(1)}$ is the occupation time in j of the first excursion of (Z_t), and we obtain (6.4) from the decomposition $O_j^{(n)} = O_j^{(1)} + (O_j^{(n)} - O_j^{(1)})$ and the strong Markov property; h_{ik} being the probability that there is a resurrection with an attendant return to k.

We note first that, by induction, $G_{ij}^{(n)} < \infty$ and then by recursion,

$$G_{ij}^{(n)} = \sum_{0\leq\nu<n} \sum_{k\in\mathcal{J}} h_{ik}^{(\nu)} \rho_{kj}.$$

Equation (6.3) follows on observing that $G_{ij}^{(n)} \nearrow G_{ij}$, and from monotone convergence. □

In summary then, for the typical case we have the following dichotomy:

(i) $P_i(T_H = \infty) > 0$ and $\mathcal{T}$ is Q-transient; or

(ii) $P_i(T_H = \infty) = 1$ $(i \in \mathcal{T})$ and $\mathcal{J}$ is $\mathcal{H}$-recurrent if the conditions of Theorem 6.1 are satisfied for all $a \in \mathcal{J}$. In addition, if $\mathcal{T}$ is $\mathcal{U}$-irreducible then it is Q-recurrent.

The last assertion follows from (6.3) since $\eta_{ia} = \infty$ for all i and $\rho_{ij} > 0$.

We now discuss the limiting behaviour of the $p_{ij}(t)$ under the following conditions. Assume that $\mathcal{T}$ is Q-recurrent and irreducible; the latter follows from our general assumptions about $\mathcal{J}$ and $\mathcal{U}$-irreducibility of $\mathcal{T}$. We suppose that $\mathcal{J}$ is $\mathcal{H}$-recurrent, ensuring a unique solution of (4.8), and let $\{\nu_i\}$ denote the stationary measure of $\mathcal{H}$; it is unique up to a constant factor. Assumption I and $\mathcal{H}$ stochastic ensures the relation $\sum_{j\in\mathcal{T}} r_{ij}(t) = P_i(T_H > t)$, and hence that

$$m_i = E_i(T_H) = \sum_{j\in\mathcal{T}} \rho_{ij}.$$

This may not be finite. Finally let

$$c = 1 \Big/ \sum_{i \in \mathcal{J}} \nu_i m_i \quad \text{and} \quad \pi_j = c \sum_{i \in \mathcal{J}} \nu_i \rho_{ij} \equiv c\phi_j \tag{6.5}$$

where $c = 0$ if the denominator is infinite, and we always assume $\phi_j < \infty$.

We expect under suitable regularity conditions that

$$\lim_{t \to \infty} p_{ij}(t) = \pi_j \quad (j \in \mathcal{T}) \tag{6.6}$$

and in particular $\mathcal{T}$ is Q-positive recurrent if $c > 0$ and Q-null if $c = 0$. Regularity conditions can be extracted from general theory, or tailored to fit the situation at hand. The latter usually gives finer results.

The following general condition comes from Athreya, McDonald and Ney's (1978) (see p. 794) account of Kesten's (1974) renewal theorem. Also see Çinlar (1975, pp. 332–334), for a slightly weaker form.

Lemma 6.1. *Assume in addition to the above conditions that*

$$\sum_{i \in \mathcal{J}} \nu_i \sum_{n \geq 0} \sup_{t \in [n, n+1]} r_{ij}(t) < \infty. \tag{6.7}$$

Then (6.6) *holds and* $c > 0$ *is necessary and sufficient for Q-positive recurrence.* □

Proposition 4.1 of Athreya, McDonald and Ney (1978) holds under the conditions of this lemma; their (i) follows from (6.7) and non-negativity, and $r_{ij}(t) \leq P_i(T_H > t) \to 0$ $(t \to \infty)$. They assume $c > 0$, but the key renewal theorem is still applicable if $c = 0$ but all the other assumptions are satisfied. The Q-null assertion above follows from this observation.

In nearly all applications we meet only minimal processes and then conditions like (6.7) are superfluous.

Theorem 6.3. *Suppose*

(i) $[r_{ij}(t)]$ *satisfies the* $\mathcal{U}$*-forward system*;

(ii) $\mathcal{T}$ *is Q-irreducible*;

(iii) $\mathcal{J}$ *is* $\mathcal{H}$*-irreducible and recurrent*;

(iv) *The equations* $\sum_{i \in \mathcal{T}} y_i q_{ij} = \lambda y_j$ *have no nontrivial solution* y_j *for some (and then for all)* $\lambda > 0$; *and*

(v) *the* ϕ_i *are finite* $(i \in \mathcal{T})$; *remember that* $\{\nu_i\}$ *is the* $\mathcal{H}$*-invariant measure.*

Then the measure $\{\phi_i\}$ *is Q-invariant, and* $\mathcal{T}$ *is Q-positive recurrent iff c (defined at* (6.5)) *is positive. In this case* $\{c\phi_i\}$ *is the LSL of* (X_t).

Proof. Condition (iii) ensures (4.8) has a unique solution, and (i), via Theorem 4.1(iii), ensures that this solution, $[p_{ij}(t)]$, is Q-minimal and that Q is regular.

Integration of the $\mathcal{U}$-forward system yields the identity

$$\sum_{k \in \mathcal{T}} \rho_{ik} u_{kj} = -\delta_{ij} \quad (i, j \in \mathcal{T}).$$

In terms of our present notation (5.3) becomes $h_{ij}=\sum_{k\in\mathcal{T},l\in H}\rho_{ik}u_{kl}p(l,j)$ whence, from (4.1),

$$\sum_{i\in\mathcal{T}}\phi_i q_{ij}=\sum_{i\in\mathcal{J},k\in\mathcal{T}}\nu_i\rho_{ik}u_{kj}+\sum_{i\in\mathcal{J},k\in\mathcal{T},l\in H}\nu_i\rho_{ik}u_{kl}p(l,j)$$

$$=-\sum_{i\in\mathcal{J}}\nu_i\delta_{ij}+\sum_{i\in\mathcal{J}}\nu_i h_{ij}\equiv 0,$$

from the above identities. Hence $\{\phi_i\}$ is an invariant measure for the matrix Q, whence from condition (iv) above, it is invariant for $[p_{ij}(t)]$ (see Anderson, 1991, p. 195), that is, $\{\phi_i\}$ is Q-invariant. The remaining assertions follow. □

We end this section with some illustrations of these theorems.

Example 6.1. The above general theory yields Theorem 3.1. To see this, recall that (J_n) is an i.i.d. sequence with law $(h_j: j\in\mathbb{N})$, whence $\nu_i=h_i$. The calculations following (3.3) (or see Pakes, 1979, p. 288) show that

$$\sum_{j\geq 1}\rho_{ij}s^j=\int_0^s(1-v^i)/a(v)\,\mathrm{d}v,$$

whence

$$\sum_{j\geq 1}\phi_j s^j=\int_0^s(1-h(v))/a(v)\,\mathrm{d}v<\infty$$

and c is given by (3.1). The assertions of Theorem 3.1 follows.

It appears that condition (v) above needs to be checked in each case, but for those applications where $\mathcal{T}=\mathbb{N}$ and $H=\{0\}$ it often is the case that ρ_{ij} is constant for $i\geq j$, j fixed. This occurs when (Z_t) is skip-free to the left, as it is for the MBP or a birth-death process. We give the following particular case of the latter where the jump chain is a random walk.

Example 6.2. Let (Z_t) be the birth and death process with parameters $\lambda_j=\alpha q_j$, $\mu_j=\beta q_j$ $(j\in\mathbb{N}_+)$ where $\alpha+\beta=1$ and $0<\alpha<\frac{1}{2}$, $q_0=0$, and $q_j>0$ if $j>0$. Then (Z_t) hits 0 in finite expected time. Let $r=\alpha/\beta$. It can be shown that

$$(\beta-\alpha)\rho_{ij}=q_j^{-1}[r^{(j-i)\vee 0}-r^j]\quad(i,j\geq 1),$$

and this is independent of i when $i\geq j$. Let $\eta_j=\sum_{i>j}h_i$. Then

$$(\beta-\alpha)/c=\sum_{j\geq 1}\eta_j/q_j+\sum_{i\geq 1}h_i\sum_{j\geq i}r^{j-i}/q_j-\sum_{j\geq 1}r^j/q_j.$$

This can be finite or infinite, depending on a delicate interplay between r, $\{h_i\}$ and $\{q_i\}$. Roughly speaking, if $\eta_i>r^i$ then $c>0$ iff $\sum\eta_j/q_j<\infty$, but if $\eta_i<r^i$ then $c>0$ iff $\sum r^j/q_j<\infty$.

We end with an example where (Z_t) is not skip-free to the left and the ρ_{ij} do not have the above constancy property. However the series defining the ϕ_i converge.

Example 6.3. We let (Z_t) be the linear birth and death process allowing catastrophes, as described by Pakes (1987). The key observation here is that $r_{ij}(t) = iq_{ij}(t)/j$ $(i, j \geq 1)$ where the $q_{ij}(t)$ are transition functions of a MBP as defined above (op. cit, p. 311). There is a single absorbing state 0 for (Z_t) and $E_i(T_H) < \infty$ if this dual MBP is supercritical.

Clearly $\rho_{ij} = i\gamma_{ji}/j$ where γ_{ij} is a Green function of the dual MBP. Following Pakes (1979, p. 290), it is quite easy to show that

$$\gamma_{ij} = (q^{i-1}/\nu j) \sum_{k<j} q^{-k} u_{j-1-k} I\{k < i\} \tag{6.8}$$

where $\{u_j\}$ is the renewal sequence induced by the law whose p.g.f. is $w(s) = (f(s) - q)/(s - q)$. We see that ρ_{ij} is not constant for $i > j$, but the discrete renewal theorem gives

$$\lim_{i\to\infty} \rho_{ij} = (1 - q^j)/\nu(m-1)$$

whence the ϕ_j are finite.

An intermediate step on the route to (6.8) gives the generating function

$$\sum_{i\geq 1} s^i \rho_{ij} = (s/j)(q^j - s^j)/\nu(f(s) - s)$$

and hence if, for example, $h_i = (1-r)r^{i-1}$ then the resurrected process (X_t) is positive recurrent and

$$\pi_j = [\log((1-r)/(1-q))]^{-1}[q^j - r^j]/j \quad (j \geq 1)$$

if $r \neq q$, and $\pi_j = (1-q)q^{j-1}$ if $r = q$.

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